

MULTI-PEAK SOLUTIONS FOR NONLINEAR CHOQUARD EQUATION WITH A GENERAL NONLINEARITY

MINBO YANG, JIANJUN ZHANG, AND YIMIN ZHANG

ABSTRACT. In this paper, we study a class of nonlinear Choquard type equations involving a general nonlinearity. By using the method of penalization argument, we show that there exists a family of solutions having multiple concentration regions which concentrate at the minimum points of the potential V . Moreover, the *monotonicity* of $f(s)/s$ and the so-called *Ambrosetti-Rabinowitz* condition are not required.

1. INTRODUCTION

This paper is concerned with the following nonlinear Choquard equation

$$-\varepsilon^2 \Delta v + V(x)v = \varepsilon^{-\alpha} (I_\alpha * F(v))f(v), \quad v \in H^1(\mathbb{R}^N), \quad (1.1)$$

where $N \geq 3$, $\alpha \in (0, N)$, F is the prime function of f and I_α is the Riesz potential defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) := \frac{\Gamma((N-\alpha)/2)}{\Gamma(\alpha/2)\pi^{N/2}2^\alpha|x|^{N-\alpha}}.$$

In the sequel, we assume that the potential function V satisfies the following conditions:

- (V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^N} V(x) = 1$.
- (V2) There are bounded disjoint open sets $O^i, i = 1, 2, \dots, k$, such that for any $i \in \{1, 2, \dots, k\}$,

$$0 < m_i \equiv \inf_{x \in O^i} V(x) < \min_{x \in \partial O^i} V(x),$$

and $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies

- (F1) $\lim_{t \rightarrow 0^+} f(t)/t = 0$;
- (F2) $\lim_{t \rightarrow +\infty} f(t)/t^{\frac{\alpha+2}{N-2}} = 0$;
- (F3) there exists $s_0 > 0$ such that $F(s_0) > 0$.

For any $i \in \{1, 2, \dots, k\}$, let

$$\mathcal{M}^i \equiv \{x \in O^i : V(x) = m_i\}.$$

The main theorem of this paper reads as

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Theorem 1.1. *Suppose that $\alpha \in ((N-4)_+, N)$, (V1)-(V2) and (F1)-(F3). Then, for sufficiently small $\varepsilon > 0$, (1.1) admits a positive solution v_ε , which satisfies*

(i) *there exist k local maximum points $x_\varepsilon^i \in O^i$ of v_ε such that*

$$\lim_{\varepsilon \rightarrow 0} \max_{1 \leq i \leq k} \text{dist}(x_\varepsilon^i, \mathcal{M}^i) = 0,$$

and $w_\varepsilon(x) \equiv v_\varepsilon(\varepsilon x + x_\varepsilon^i)$ converges (up to a subsequence) uniformly to a least energy solution of

$$-\Delta u + m_i u = (I_\alpha * F(u))f(u), \quad u \in H^1(\mathbb{R}^N); \quad (1.2)$$

(ii) *$v_\varepsilon(x) \leq C \exp(-\frac{c}{\varepsilon} \min_{1 \leq i \leq k} |x - x_\varepsilon^i|)$ for some $c, C > 0$.*

Our motivation for the study of such a problem goes back at least to the pioneering work of Floer and Weinstein [17] (see also [33]) concerning the Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^N. \quad (1.3)$$

By means of a Lyapunov-Schmidt reduction approach, these authors constructed single-peak or multi-peak solutions of (1.3) concentrating around any given non-degenerate critical points of V as $\varepsilon \rightarrow 0$. For $\varepsilon > 0$ sufficiently small, these standing waves are referred to as semi-classical states, which describe the transition from quantum mechanics to classical mechanics. For the detailed physical background, we refer to [33] and references therein. In [17, 33], their arguments are based on a Lyapunov-Schmidt reduction in which a non-degenerate condition plays a crucial role. Without such a non-degenerate condition, by using the mountain pass argument, Rabinowitz [37] proved the existence of positive solutions of (1.3) for small $\varepsilon > 0$ provided the following global potential well condition

$$\liminf_{|x| \rightarrow \infty} V(x) > \inf_{\mathbb{R}^N} V(x)$$

holds. Subsequently, by virtue of a penalization approach, del Pino and Felmer [13] established the existence of a single-peak solution to (1.3) which concentrates around local minimum points of V . Some related results can be found in [1, 14–16, 31, 41] and the references therein. In the works above, the nonlinearity f satisfies the monotonicity condition

$$f(s)/|s| \text{ is strictly increasing for } s \neq 0 \quad (\text{N})$$

or the well-known *Ambrosetti-Rabinowitz condition*

$$0 < \mu \int_0^s f(t) dt \leq s f(s) \text{ for any } t \neq 0 \text{ and some } \mu > 2. \quad (\text{AR})$$

To attack the existence of positive solutions to (1.3) without (N) and (AR), by introducing a new penalization approach, Byeon and Jeanjean [5] constructed a spike solution near local minimal points of V under a almost optimal hypotheses:

(BL1) $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{t \rightarrow 0} f(t)/t = 0$;

(BL2) there exists $p \in (1, (N+2)/(N-2))$ such that $\limsup_{t \rightarrow \infty} f(t)/t^p < \infty$;

(BL3) there exists $T > 0$ such that $mT^2/2 < F(T) \equiv \int_0^T f(t) dt$.

(BL1)-(BL3) are referred to as the Berestycki-Lions conditions, which were firstly proposed by a celebrated paper [4]. We refer the reader to [7–9] and the references therein for the development on this subject.

Taking $u(x) = v(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$, then (1.1) is equivalent to the following problem

$$-\Delta u + V_\varepsilon(x)u = (I_\alpha * F(u))f(u), \quad u \in H^1(\mathbb{R}^N). \quad (1.4)$$

Obviously, the term $(I_\alpha * F(v))f(v)$ is nonlocal. Equation (1.4) can be seen as a special case of the generalized nonlocal Schrödinger equation

$$-\Delta u + V_\varepsilon(x)u = (K(x) * F(u))f(u), \quad u \in H^1(\mathbb{R}^N). \quad (1.5)$$

From the view of physical background, $K(x)$ is called as a response function which possesses the information on the mutual interaction between the bosons. In general, the following equation for $a > 0$ is considered as the limiting equation of (1.4)

$$-\Delta u + au = (I_\alpha * F(u))f(u), \quad u \in H^1(\mathbb{R}^N). \quad (1.6)$$

For $N = 3$, $\alpha = 2$ and $f(s) = s$, (1.1) and (1.6) reduce to

$$-\varepsilon^2 \Delta v + V(x)v = \varepsilon^{-2}(I_2 * v^2)v/2, \quad x \in \mathbb{R}^3 \quad (1.7)$$

and

$$-\Delta u + au = (I_2 * u^2)u/2, \quad x \in \mathbb{R}^3. \quad (1.8)$$

Equation (1.8) is commonly named as the stationary Choquard equation. In 1976, during the symposium on Coulomb systems at Lausanne, Choquard proposed this type of equations as an approximation to Hartree-Fock theory for a one component plasma [22]. It arises in multiple particle systems [19, 22], quantum mechanics [34–36] and laser beams, etc. In the recent years, There has been a considerable attention to be paid on investigating the Choquard equation. In the pioneering works [20], Lieb investigated the existence and uniqueness of positive solutions to equation (1.8). Subsequently, Lions [23, 24] obtained the existence and multiplicity results for (1.8) via the critical point theory. In [26], Ma and Zhao studied the classification of all positive solutions to the nonlinear Choquard problem

$$-\Delta u + u = (|x|^{-\alpha} * |u|^p)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N), \quad (1.9)$$

where $\alpha \in (0, N)$ and $p \in [2, (2N - \alpha)/(N - 2))$. Due to the present of the nonlocal term, the standard method of moving planes cannot be used directly. So the classification of positive solutions to (1.9)(even for $p = 2$) had remained as an longstanding open problem. By using the integral form of the method of moving planes introduced by Chen et al. [11], Ma and Zhao [26] solved this open problem. Precisely, they proved that up to translations, positive solutions of equation (1.9) are radially symmetric and monotone decreasing, under some assumption on α , p and N . In [30], Moroz and van Schaftingen eliminated this restriction and established an optimal range of parameters for the existence of a positive ground state solution of (1.9). Moreover, they proved that all positive ground state solutions of (1.9) are radially symmetric and monotone decaying about some point. Later, in the spirit of Berestycki and Lions, Moroz and van Schaftingen [28] gave a almost necessary conditions on the nonlinearity f for the

existence of ground state solutions of (1.4). The symmetry of solutions was considered in [28] as well.

In the present paper, we are interested in semiclassical state solutions of (1.1). For the special case (1.7), there have been many results on this subject(see [12, 27, 32, 38, 40] and the references therein). By using a Lyapunov-Schmidt reduction argument, Wei and Winter [40] proved the existence of multibump solutions of (1.7) concentrating at local minima, local maxima or non-degenerate critical points of V provided $\inf V > 0$. Subsequently, Secchi [38] studied the case of the potential $V > 0$ and satisfying $\liminf_{|x| \rightarrow \infty} V(x)|x|^\gamma > 0$ for some $\gamma \in [0, 1)$. By a perturbation technique, they obtained the existence of positive bound state solution concentrating at local minimum (or maximum) points of V when $\varepsilon \rightarrow 0$. Moroz and Van Schaftingen [29] considered the semiclassical states of the Choquard equation (1.1) with $f(s) = |s|^{p-2}s$, $p \in [2, (N + \alpha)/(N - 2)_+)$. By introducing a novel nonlocal penalization technique, the authors proved that (1.1) has a family of solutions concentrating at the local minimum of V . Moreover, in [29] the potential V maybe vanishes at infinity, and the assumptions on the decay of V and the admissible range for $p \geq 2$ are optimal. In [42], Yang and Ding considered the following equation

$$-\varepsilon^2 \Delta u + V(x)u = \left[\frac{1}{|x|^\mu} * u^p \right] u^{p-1}, \quad \text{in } \mathbb{R}^3. \quad (1.10)$$

By using the variational methods, for suitable parameters p, μ , the authors obtained the existence of solutions of (1.10). By the penalization method in [13], Alves and Yang [3] considered the concentration behavior of solutions to the following generalized quasilinear Choquard equation

$$-\varepsilon^p \Delta_p v + V(x)|v|^{p-2}v = \varepsilon^{\mu-N} \left(\int_{\mathbb{R}^N} \frac{Q(y)F(v(y))}{|x-y|^\mu} dy \right) Q(x)f(v), \quad x \in \mathbb{R}^N,$$

where Δ_p is the p -Laplacian operator, $p \in (1, N)$ and $\mu \in (0, N)$. For the related results, we refer to [2, 12, 39] and the references therein.

To sum up, in all the works mentioned above, the authors only considered the Choquard equation (1.1) with a power type nonlinearity or a general nonlinearity satisfying some sort of monotonicity condition or Ambrosetti-Rabinowitz type condition. Similar to [5] for the local problem (1.3), it seems natural to ask

Does the similar concentration phenomenon occur for the Choquard equation (1.1) under very mild assumptions on f in the spirit of Berestycki and Lions?

In the present paper, we will give an affirmative answer to this question. In particular, the monotonicity condition and Ambrosetti-Rabinowitz condition are not required.

The spirit of this paper is somewhat akin to [5, 6]. The penalization argument is used to prove Theorem 1.1. This method is widely used by many authors. The penalization functional we need was first introduced by Byeon and Wang in [10].

2. PROOF OF THEOREM 1.1

In this section, we will use the framework of Byeon and Jeanjean [6](see also [5]) to prove our main result.

2.1. The limit problem. We define an energy functional for the limiting problem (1.6) by

$$L_a(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + au^2 - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u), \quad u \in H^1(\mathbb{R}^N).$$

Let E_a be the least energy of (1.6) and S_a be the set of least energy solutions U of (1.6) satisfying $U(0) = \max_{x \in \mathbb{R}^N} U(x)$, the following property of S_a was proved in [28].

Proposition 2.1. [28] *Assume that f satisfies (F1)-(F3), then*

- (i) $S_a \neq \emptyset$ and S_a is compact in $H^1(\mathbb{R}^N)$.
- (ii) E_a coincides with the mountain pass value.
- (iii) For any $U \in S_a$, $U \in W_{loc}^{2,q}(\mathbb{R}^N)$ for any $q \geq 1$. Moreover,
- (iv) U is radially symmetric and radially decreasing.
- (v) U satisfies the Pohožev identity:

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla U|^2 + \frac{N}{2} a \int_{\mathbb{R}^N} U^2 = \frac{N+\alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(U))F(U).$$

Now, we give some further estimates about the boundedness and decay for any $U \in S_a$. The following Hardy-Littlewood-Sobolev inequality will be used frequently later.

Lemma 2.1. [21, Theorem 4.3] *Let $s, r > 1$ and $0 < \alpha < N$ with $1/s + 1/r = 1 + \alpha/N$, $f \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a constant $C(s, N, \alpha, r)$ (independent of f, h) such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x) |x-y|^{N-\alpha} g(y) dx dy \right| \leq C(s, N, \alpha) \|f\|_s \|g\|_r,$$

where the sharp constant $C(s, N, \alpha)$ satisfies

$$C(s, N, \alpha) \leq \frac{N}{sr\alpha} (|\mathbb{S}^{N-1}|/N)^{1-\alpha/N} \left[\left(\frac{1-\alpha/N}{1-1/s} \right)^{1-\alpha/N} + \left(\frac{1-\alpha/N}{1-1/r} \right)^{1-\alpha/N} \right].$$

Now, we adopt some ideas from [2, 28] to give the decay of the ground state solutions to (1.6).

Proposition 2.2. S_a is uniformly bounded in $L^\infty(\mathbb{R}^N)$. Moreover, there exist $C, c > 0$, independent of $u \in S_a$, such that $|D^{\alpha_1} u(x)| \leq C \exp(-c|x|)$, $x \in \mathbb{R}^N$ for $|\alpha_1| = 0, 1$.

Proof. First, we give the uniformly boundedness of $u \in S_a$. For any $u \in S_a$, we get

$$L_a(u) = \frac{2+\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\alpha a}{2(N+\alpha)} \int_{\mathbb{R}^N} u^2 = E_a,$$

which implies that S_a is bounded in $H^1(\mathbb{R}^N)$. Let $H(u) = F(u)/u$ and $K(u) = f(u)$, then by (F1)-(F2), there exists $C > 0$ (independent of u) such that

$$|H(u(x))|, |K(u(x))| \leq C (|u(x)|^{\alpha/N} + |u(x)|^{(\alpha+2)/(N-2)}), \quad x \in \mathbb{R}^N.$$

It follows that $H(u), K(u) \in L^{2N/\alpha}(\mathbb{R}^N) + L^{2N/(\alpha+2)}(\mathbb{R}^N)$, i. e., $H(u) = H^*(u) + H_*(u)$, $K(u) = K^*(u) + K_*(u)$ with $H^*(u), K^*(u) \in L^{2N/\alpha}(\mathbb{R}^N)$ and $H_*(u), K_*(u) \in L^{2N/(\alpha+2)}(\mathbb{R}^N)$. Moreover, $H^*(u), K^*(u)$ are uniformly bounded in $L^{2N/\alpha}(\mathbb{R}^N)$ for any $u \in S_a$. So is $H_*(u), K_*(u)$ in $L^{2N/(\alpha+2)}(\mathbb{R}^N)$. Then by [28, Proposition 3.1], for any

$u \in S_a$ we get $u \in L^p(\mathbb{R}^N)$ for $p \in [2, \frac{N}{\alpha} \frac{2N}{N-2})$. Meanwhile, there exists C_p (depending on only p) such that for any $p \in [2, \frac{N}{\alpha} \frac{2N}{N-2})$,

$$\|u\|_p \leq C_p \|u\|_2, \text{ for all } u \in S_a. \quad (2.1)$$

Now, we claim that $I_\alpha * F(u)$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$ for $u \in S_a$. By (F1)-(F2), there exists $c > 0$ such that $|F(\tau)| \leq C(|\tau|^2 + |\tau|^{(N+\alpha)/(N-2)})$ for all $\tau \in \mathbb{R}$. Then for any $x \in \mathbb{R}^N$ and $u \in S_a$, there exists $C(\alpha)$ (depending only N, α) such that

$$\begin{aligned} (I_\alpha * |F(u)|)(x) &\leq C(\alpha) \int_{|x-y| \geq 1} \frac{|u|^2 + |u|^{(N+\alpha)/(N-2)}}{|x-y|^{N-\alpha}} dy \\ &\quad + C(\alpha) \int_{|x-y| \leq 1} \frac{|u|^2 + |u|^{(N+\alpha)/(N-2)}}{|x-y|^{N-\alpha}} dy \\ &\leq C(\alpha) \int_{\mathbb{R}^2} (|u|^2 + |u|^{(N+\alpha)/(N-2)}) dy \\ &\quad + C(\alpha) \int_{|x-y| \leq 1} \frac{|u|^2 + |u|^{(N+\alpha)/(N-2)}}{|x-y|^{N-\alpha}} dy. \end{aligned}$$

By $\alpha \geq N-4$, $(N+\alpha)/(N-2) \in [2, 2N/(N-2))$. By (2.1), there exists c (independent of u) such that for any $x \in \mathbb{R}^N$,

$$(I_\alpha * |F(u)|)(x) \leq C + C(\alpha) \int_{|x-y| \leq 1} \frac{|u|^2 + |u|^{(N+\alpha)/(N-2)}}{|x-y|^{N-\alpha}} dy.$$

In the following, we estimate the term

$$\int_{|x-y| \leq 1} \frac{|u|^2 + |u|^{(N+\alpha)/(N-2)}}{|x-y|^{N-\alpha}} dy.$$

Choosing $t \in (\frac{N}{\alpha}, \frac{N}{\alpha} \frac{2N}{N-2})$ with $2t \in (2, \frac{N}{\alpha} \frac{2N}{N-2})$ and $(\alpha - N)(t/(t-1)) + N > 0$,

$$\begin{aligned} &\int_{|x-y| \leq 1} |x-y|^{\alpha-N} u^2 dy \\ &\leq \|u\|_{2t}^2 \left(\int_{|x-y| \leq 1} |x-y|^{(\alpha-N)(t/(t-1))} dy \right)^{1-1/t} \leq C_1 \|u\|_{2t}^2. \end{aligned}$$

Choosing $s \in (\frac{N}{\alpha}, \frac{N}{\alpha} \frac{2N}{N-2})$ with $s \frac{N+\alpha}{N-2} \in (2, \frac{N}{\alpha} \frac{2N}{N-2})$ and $(\alpha - N)(s/(s-1)) + N > 0$,

$$\begin{aligned} &\int_{|x-y| \leq 1} |x-y|^{\alpha-N} |u|^{(N+\alpha)/(N-2)} dy \\ &\leq \|u\|_{s \frac{N+\alpha}{N-2}}^{(N+\alpha)/(N-2)} \left(\int_{|x-y| \leq 1} |x-y|^{(\alpha-N)(s/(s-1))} dy \right)^{1-1/s} \\ &\leq C_2 \|u\|_{s \frac{N+\alpha}{N-2}}^{(N+\alpha)/(N-2)}. \end{aligned}$$

Thus by (2.1) $I_\alpha * |F(u)|$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$ for all $u \in S_a$. By the standard Moser iteration, S_a is uniformly bounded in $L^\infty(\mathbb{R}^N)$. Moreover, by the radial lemma, one knows $u(x) \rightarrow 0$ uniformly as $|x| \rightarrow \infty$ for $u \in S_a$. By virtue of the comparison principle,

there exist $C, c > 0$, independent of $u \in S_a$, such that $|D^{\alpha_1} u(x)| \leq C \exp(-c|x|)$, $x \in \mathbb{R}^N$ for $|\alpha_1| = 0, 1$. \square

2.2. The penalization argument. To study (1.1), it suffices to investigate (1.4). Let H_ε be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V_\varepsilon u^2) \right)^{\frac{1}{2}}.$$

Since $\inf_{\mathbb{R}^N} V(x) = 1$, $H_\varepsilon \subset H^1(\mathbb{R}^N)$. For any set $B \subset \mathbb{R}^N$ and $\varepsilon > 0$, we define $B_\varepsilon \equiv \{x \in \mathbb{R}^N : \varepsilon x \in B\}$ and $B^\delta \equiv \{x \in \mathbb{R}^N : \text{dist}(x, B) \leq \delta\}$. Let $\mathcal{M} = \cup_{i=1}^k \mathcal{M}^i$ and $O = \cup_{i=1}^k O^i$. Since we are interested in the positive solutions of (1.1), from now on, we may assume that $f(t) = 0$ for $t \leq 0$. For $u \in H_\varepsilon$, let

$$P_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V_\varepsilon u^2 - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u).$$

Fixing an arbitrary $\mu > 0$, we define

$$\chi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in O_\varepsilon, \\ \varepsilon^{-\mu}, & \text{if } x \in \mathbb{R}^N \setminus O_\varepsilon, \end{cases} \quad \chi_\varepsilon^i(x) = \begin{cases} 0, & \text{if } x \in O_\varepsilon^i, \\ \varepsilon^{-\mu}, & \text{if } x \in \mathbb{R}^N \setminus O_\varepsilon^i, \end{cases}$$

and

$$Q_\varepsilon(u) = \left(\int_{\mathbb{R}^N} \chi_\varepsilon u^2 dx - 1 \right)_+^2, \quad Q_\varepsilon^i(u) = \left(\int_{\mathbb{R}^N} \chi_\varepsilon^i u^2 dx - 1 \right)_+^2.$$

Let $\Gamma_\varepsilon, \Gamma_\varepsilon^i (i = 1, 2, \dots, k) : H_\varepsilon \rightarrow \mathbb{R}$ be given by

$$\Gamma_\varepsilon(u) = P_\varepsilon(u) + Q_\varepsilon(u), \Gamma_\varepsilon^i(u) = P_\varepsilon(u) + Q_\varepsilon^i(u).$$

It is standard to check that $\Gamma_\varepsilon, \Gamma_\varepsilon^i \in C^1(H_\varepsilon)$. To find solutions of (1.4) which concentrate in O as $\varepsilon \rightarrow 0$, we shall search critical points of Γ_ε such that Q_ε is zero. The functional Q_ε that was first introduced in [10], will act as a penalization to force the concentration phenomena to occur inside O .

Now, we construct a set of approximate solutions of (1.4). Let

$$\delta = \frac{1}{10} \min\{\text{dist}(\mathcal{M}, O^c), \min_{i \neq j} \text{dist}(O^i, O^j)\}.$$

We fix a $\beta \in (0, \delta)$ and a cut-off $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $|x| \leq \beta$ and $\varphi(x) = 0$ for $|x| \geq 2\beta$. Let $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$, $y \in \mathbb{R}^N$ and for some $x_i \in (\mathcal{M}^i)^\beta$, $1 \leq i \leq k$, and $U_i \in S_{m_i}$, we define

$$U_\varepsilon^{x_1, x_2, \dots, x_k}(y) = \sum_{i=1}^k \varphi_\varepsilon\left(y - \frac{x_i}{\varepsilon}\right) U_i\left(y - \frac{x_i}{\varepsilon}\right).$$

As in [6], we will find a solution near the set

$$X_\varepsilon = \{U_\varepsilon^{x_1, x_2, \dots, x_k} \mid x_i \in (\mathcal{M}^i)^\beta, U_i \in S_{m_i}, i = 1, 2, \dots, k\}$$

for sufficiently small $\varepsilon > 0$. For each $1 \leq i \leq k$, Choosing some $U_i \in S_{m_i}$ and $x_i \in \mathcal{M}^i$ but fixed, define

$$W_{\varepsilon, t}^i(\cdot) \equiv (\varphi_\varepsilon U_{i, t})(\cdot - x_i/\varepsilon) \text{ where } U_{i, t}(\cdot) = U_i(\cdot/t).$$

Lemma 2.2. *There exist $T_i > 0, i = 1, 2, \dots, k$, such that for $\varepsilon > 0$ small enough, $\Gamma_\varepsilon(W_{\varepsilon, T_i}^i) < -2, i = 1, 2, \dots, k$.*

Proof. By a direct calculation, we get $\Gamma_\varepsilon(W_{\varepsilon, t}^i) = P_\varepsilon(W_{\varepsilon, t}^i)$ for any $t > 0$ and for any $1 \leq i \leq k$,

$$L_{m_i}(U_{i, t}) = \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla U_i|^2 + \frac{t^N}{2} m_i \int_{\mathbb{R}^N} |U_i|^2 - \frac{t^{N+\alpha}}{2} \int_{\mathbb{R}^N} (I_\alpha * F(U_i)) F(U_i).$$

Then there exists $T_i > 1$ such that $L_{m_i}(U_{i, T_i}) < -2$ for $t > T_i$. Notice that

$$P_\varepsilon(W_{\varepsilon, t}^i) = L_{m_i}(W_{\varepsilon, t}^i) + \frac{1}{2} \int_{\mathbb{R}^N} (V_\varepsilon(x) - m_i)(W_{\varepsilon, t}^i)^2,$$

we have

$$P_\varepsilon(W_{\varepsilon, t}^i) = L_{m_i}(U_{i, t}) + O(\varepsilon),$$

by the decay property of U_i in proposition 2.2. Consequently, we know that $\Gamma_\varepsilon(W_{\varepsilon, T_i}^i) < -2$ for $\varepsilon > 0$ small. \square

For any $1 \leq i \leq k$, let $\gamma_\varepsilon^i(t)(\cdot) = W_{\varepsilon, t}^i(\cdot)$ for $t > 0$. Due to $N \geq 3$, $\lim_{t \rightarrow 0} \|W_{\varepsilon, t}^i\|_\varepsilon = 0$, let $\gamma_\varepsilon^i(0) = 0$. Then as in [6], we define a min-max value C_ε^i :

$$C_\varepsilon^i = \inf_{\varphi \in \Phi_\varepsilon^i} \max_{s_i \in [0, T_i]} \Gamma_\varepsilon^i(\varphi(s_i)),$$

where $\Phi_\varepsilon^i = \{\varphi \in C([0, T_i], H_\varepsilon) : \varphi(0) = 0, \varphi(T_i) = \gamma_\varepsilon^i(T_i)\}$. Similar to Proposition 2 and 3 in [5], we have

Proposition 2.3. *For any $1 \leq i \leq k$, we have*

$$\lim_{\varepsilon \rightarrow 0} C_\varepsilon^i = E_{m_i}.$$

Finally, let

$$\gamma_\varepsilon(s) = \sum_{i=1}^k \gamma_\varepsilon^i(s_i), \quad s = (s_1, s_2, \dots, s_k)$$

and

$$D_\varepsilon = \max_{s \in T} \Gamma_\varepsilon(\gamma_\varepsilon(s)),$$

where $T \equiv [0, T_1] \times \dots \times [0, T_k]$. Since $\text{supp}(\gamma_\varepsilon(s)) \subset \mathcal{M}_\varepsilon^\beta$ for each $s \in T$, it follows that

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) = P_\varepsilon(\gamma_\varepsilon(s)) = \sum_{i=1}^k P_\varepsilon(\gamma_\varepsilon^i(s)).$$

By the Pohožev's identity, for any $1 \leq i \leq k$, we have

$$L_{m_i}(U_{i, t}) = \left(\frac{t^{N-2}}{2} - \frac{N-2}{N+\alpha} \frac{t^{N+\alpha}}{2} \right) \int_{\mathbb{R}^N} |\nabla U_i|^2 + \left(\frac{t^N}{2} - \frac{N}{N+\alpha} \frac{t^{N+\alpha}}{2} \right) m_i \int_{\mathbb{R}^N} |U_i|^2.$$

Let

$$g_1(t) = \frac{t^{N-2}}{2} - \frac{N-2}{N+\alpha} \frac{t^{N+\alpha}}{2}, \quad g_2(t) = \frac{t^N}{2} - \frac{N}{N+\alpha} \frac{t^{N+\alpha}}{2},$$

then it is easy to know $g'_j(t) > 0$ for $t \in (0, 1)$ and $g'_j(t) < 0$ for $t > 1$, $j = 1, 2$. Thus, for any $1 \leq i \leq k$, $L_{m_i}(U_{i,t})$ achieves a unique maximum point at $t = 1$ for $t > 0$, i. e.,

$$\max_{t>0} L_{m_i}(U_{i,t}) = L_{m_i}(U_i) = E_{m_i},$$

which leads to the following conclusion.

Proposition 2.4.

- (i) $\lim_{\varepsilon \rightarrow 0} D_\varepsilon = \sum_{i=1}^k E_{m_i} := E$;
- (ii) $\limsup_{\varepsilon \rightarrow 0} \max_{s \in \partial T} \Gamma_\varepsilon(\gamma_\varepsilon(s)) \leq \tilde{E}$, where $\tilde{E} = \max_{1 \leq j \leq k} (\sum_{i \neq j} E_{m_i})$;
- (iii) for any $d > 0$, there exists $\alpha_0 > 0$ such that for $\varepsilon > 0$ small,

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) \geq D_\varepsilon - \alpha_0 \text{ implies that } \gamma_\varepsilon(s) \in X_\varepsilon^{d/2}.$$

Now define

$$\Gamma_\varepsilon^\alpha := \{u \in H_\varepsilon : \Gamma_\varepsilon(u) \leq \alpha\}$$

and for a set $A \subset H_\varepsilon$ and $\alpha > 0$, let

$$A^\alpha := \{u \in H_\varepsilon : \inf_{v \in A} \|u - v\|_\varepsilon \leq \alpha\}.$$

In the following, we will construct a special PS-sequence of Γ_ε , which is localized in some neighborhood X_ε^d of X_ε .

Proposition 2.5. Let $\{\varepsilon_j\}$ with $\lim_{j \rightarrow \infty} \varepsilon_j = 0$, $\{u_{\varepsilon_j}\} \subset X_{\varepsilon_j}^d$ be such that

$$\lim_{j \rightarrow \infty} \Gamma_{\varepsilon_j}(u_{\varepsilon_j}) \leq E \text{ and } \lim_{j \rightarrow \infty} \Gamma'_{\varepsilon_j}(u_{\varepsilon_j}) = 0. \quad (2.2)$$

Then for sufficiently small $d > 0$, there exist, up to a subsequence, $\{y_j^i\} \subset \mathbb{R}^N$, $i = 1, 2, \dots, k$, points $x^i \in \mathcal{M}^i$, $U_i \in S_{m_i}$ such that

$$\lim_{j \rightarrow \infty} |\varepsilon_j y_j^i - x^i| = 0, \quad (2.3)$$

and

$$\lim_{j \rightarrow \infty} \left\| u_{\varepsilon_j} - \sum_{i=1}^k \varphi_{\varepsilon_j}(\cdot - y_j^i) U_i(\cdot - y_j^i) \right\|_{\varepsilon_j} = 0. \quad (2.4)$$

Proof. Without confusion, we write ε for ε_j . Since S_{m_i} is compact, then there exist $Z_i \in S_{m_i}$, $x_\varepsilon^i \in (\mathcal{M}^i)^\beta$, $x^i \in (\mathcal{M}^i)^\beta$, $i = 1, 2, \dots, k$, $\lim_{\varepsilon \rightarrow 0} x_\varepsilon^i = x^i$, such that up to a subsequence, denoted still by $\{u_\varepsilon\}$ satisfying that for sufficiently small $\varepsilon > 0$,

$$\left\| u_\varepsilon - \sum_{i=1}^k \varphi_\varepsilon\left(\cdot - \frac{x_\varepsilon^i}{\varepsilon}\right) Z_i\left(\cdot - \frac{x_\varepsilon^i}{\varepsilon}\right) \right\|_\varepsilon \leq 2d. \quad (2.5)$$

Set $u_{1,\varepsilon}(x) = \sum_{i=1}^k \varphi_\varepsilon\left(x - \frac{x_\varepsilon^i}{\varepsilon}\right) u_\varepsilon$, $u_{2,\varepsilon}(x) = u_\varepsilon(x) - u_{1,\varepsilon}(x)$.

Step 1. We claim that

$$\Gamma_\varepsilon(u_\varepsilon) \geq \Gamma_\varepsilon(u_{1,\varepsilon}) + \Gamma_\varepsilon(u_{2,\varepsilon}) + O(\varepsilon). \quad (2.6)$$

Suppose that there exist $x_\varepsilon \in \bigcup_{i=1}^k B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \setminus B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{\beta}{\varepsilon}\right)$ and $R > 0$, such that

$$\liminf_{\varepsilon \rightarrow 0} \int_{B(x_\varepsilon, R)} |u_\varepsilon|^2 dx > 0. \quad (2.7)$$

Let $W_\varepsilon = u_\varepsilon(x + x_\varepsilon)$. Using (2.7), we get

$$\liminf_{\varepsilon \rightarrow 0} \int_{B(0, R)} |W_\varepsilon|^2 dx > 0. \quad (2.8)$$

Since $\varepsilon x_\varepsilon \in \bigcup_{i=1}^k B(x_\varepsilon^i, 2\beta) \setminus B(x_\varepsilon^i, \beta)$, by taking a subsequence, we can assume $\varepsilon x_\varepsilon \rightarrow x_0 \in \bigcup_{i=1}^k B(x^i, 2\beta) \setminus B(x^i, \beta)$. From (2.5), one has $\{W_\varepsilon\}$ is bounded in H_ε and $H^1(\mathbb{R}^N)$.

Without loss of generality, we assume that $W_\varepsilon \rightharpoonup W$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L_{loc}^q(\mathbb{R}^N)$ for $q \in [2, 2^*)$. Clearly, (2.8) implies that $W \neq 0$ and from (2.2) we get that W is a nontrivial solution of

$$-\Delta W + V(x_0)W = (I_\alpha * F(W))f(W) \text{ in } \mathbb{R}^N. \quad (2.9)$$

Once choosing R large enough, we deduce by the weak convergence that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{B(x_\varepsilon, R)} (|\nabla u_\varepsilon|^2 + V_\varepsilon(x)u_\varepsilon^2) dx &= \liminf_{\varepsilon \rightarrow 0} \int_{B(0, R)} (|\nabla W_\varepsilon|^2 + V_\varepsilon(x + x_\varepsilon)|W_\varepsilon|^2) dx \\ &\geq \int_{B(0, R)} (|\nabla W|^2 + V(x_0)|W|^2) dx \geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla W|^2 + V(x_0)|W|^2) dx. \end{aligned} \quad (2.10)$$

By Proposition 2.1, E_a is a mountain pass value. One can get E_a is strictly increasing for $a > 0$. Then

$$L_{V(x_0)}(W) \geq E_{V(x_0)} \geq E_m, \quad \text{since } V(x_0) \geq m.$$

Thus by (2.10) and Pohozaev identity, we get

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{B(x_\varepsilon, R)} (|\nabla u_\varepsilon|^2 + V_\varepsilon(x)u_\varepsilon^2) dx &\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla W|^2 + V(x_0)W^2) dx \\ &\geq \frac{N + \alpha}{\alpha + 2} L_{V(x_0)}(W) \geq \frac{N + \alpha}{\alpha + 2} E_{V(x_0)}(W), \end{aligned}$$

which contradicts (2.5) for $d > 0$ small enough. It follows from [25, Lemma I.1] that

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{\bigcup_{i=1}^k B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \setminus B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{\beta}{\varepsilon}\right)} |u_\varepsilon|^q dx = 0, \quad \text{for any } 2 \leq q < 2^*. \quad (2.11)$$

As a consequence, we can derive that

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_\varepsilon))F(u_\varepsilon) = \int_{\mathbb{R}^N} (I_\alpha * F(u_{1,\varepsilon}))F(u_{1,\varepsilon}) + \int_{\mathbb{R}^N} (I_\alpha * F(u_{2,\varepsilon}))F(u_{2,\varepsilon}) + o_\varepsilon(1). \quad (2.12)$$

Indeed, let $G(u_\varepsilon) := F(u_\varepsilon) - F(u_{1,\varepsilon}) - F(u_{2,\varepsilon})$, then

$$G(u_\varepsilon)(x) = 0, \text{ if } x \notin \bigcup_{i=1}^k B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \setminus B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{\beta}{\varepsilon}\right).$$

By (F1)-(F2), for any $\delta_0 > 0$ there exists $c > 0$ (depending on δ_0) such that $|G(u_\varepsilon)| \leq c|u_\varepsilon|^2 + \delta_0|u_\varepsilon|^{(N+\alpha)/(N-2)}$. Then by Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} (I_\alpha * G(u_\varepsilon)) F(u_\varepsilon) \right| \\ & \leq C(N, \alpha) \left(\int_{\mathbb{R}^N} |G(u_\varepsilon)|^{2N/(N+\alpha)} \right)^{(N+\alpha)/(2N)} \left(\int_{\mathbb{R}^N} |F(u_\varepsilon)|^{2N/(N+\alpha)} \right)^{(N+\alpha)/(2N)} \\ & \leq C'(N, \alpha) \left[\int_{\bigcup_{i=1}^k B\left(\frac{x_i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \setminus B\left(\frac{x_i}{\varepsilon}, \frac{\beta}{\varepsilon}\right)} (c|u_\varepsilon|^{4N/(N+\alpha)} + \delta_0^{2N/(N+\alpha)} |u_\varepsilon|^{2^*}) \right]^{(N+\alpha)/(2N)}. \quad (2.13) \end{aligned}$$

Recalling that $\alpha > N - 4$, $4N/(N + \alpha) \in (2, 2^*)$. By the arbitrary of δ_0 , it follows from (2.11) and (2.13) that

$$\int_{\mathbb{R}^N} (I_\alpha * G(u_\varepsilon)) F(u_\varepsilon) \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

Similarly,

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_{1,\varepsilon})) G(u_\varepsilon) \rightarrow 0, \int_{\mathbb{R}^N} (I_\alpha * F(u_{2,\varepsilon})) G(u_\varepsilon) \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

Then

$$\begin{aligned} & \int_{\mathbb{R}^N} (I_\alpha * F(u_\varepsilon)) F(u_\varepsilon) \\ & = \int_{\mathbb{R}^N} [I_\alpha * (F(u_{1,\varepsilon}) + F(u_{2,\varepsilon}) + G(u_\varepsilon))] (F(u_{1,\varepsilon}) + F(u_{2,\varepsilon}) + G(u_\varepsilon)) \\ & = \int_{\mathbb{R}^N} [I_\alpha * F(u_{1,\varepsilon})] F(u_{1,\varepsilon}) + \int_{\mathbb{R}^N} [I_\alpha * F(u_{2,\varepsilon})] F(u_{2,\varepsilon}) \\ & \quad + \int_{\mathbb{R}^N} [I_\alpha * F(u_{2,\varepsilon})] F(u_{1,\varepsilon}) + \int_{\mathbb{R}^N} [I_\alpha * F(u_{1,\varepsilon})] F(u_{2,\varepsilon}) + o_\varepsilon(1). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} [I_\alpha * F(u_{1,\varepsilon})] F(u_{2,\varepsilon}) \right| \\ & \leq \int_{\left(\bigcup_{i=1}^k B\left(\frac{x_i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right)\right) \times \left(\mathbb{R}^N \setminus \bigcup_{i=1}^k B\left(\frac{x_i}{\varepsilon}, \frac{\beta}{\varepsilon}\right)\right)} I_\alpha(x - y) |F(u_{1,\varepsilon}(x))| |F(u_{2,\varepsilon}(y))| \\ & = \int_{\left(\bigcup_{i=1}^k B\left(\frac{x_i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \setminus B\left(\frac{x_i}{\varepsilon}, \frac{\beta}{\varepsilon}\right)\right) \times \left(\mathbb{R}^N \setminus \bigcup_{i=1}^k B\left(\frac{x_i}{\varepsilon}, \frac{\beta}{\varepsilon}\right)\right)} I_\alpha(x - y) |F(u_{1,\varepsilon}(x))| |F(u_{2,\varepsilon}(y))| \\ & \quad + \int_{\bigcup_{i=1}^k B\left(\frac{x_i}{\varepsilon}, \frac{\beta}{\varepsilon}\right) \times \left(\mathbb{R}^N \setminus \bigcup_{i=1}^k B\left(\frac{x_i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right)\right)} I_\alpha(x - y) |F(u_{1,\varepsilon}(x))| |F(u_{2,\varepsilon}(y))| \\ & \quad + \int_{\bigcup_{i=1}^k B\left(\frac{x_i}{\varepsilon}, \frac{\beta}{\varepsilon}\right) \times \left(\bigcup_{i=1}^k B\left(\frac{x_i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \setminus B\left(\frac{x_i}{\varepsilon}, \frac{\beta}{\varepsilon}\right)\right)} I_\alpha(x - y) |F(u_{1,\varepsilon}(x))| |F(u_{2,\varepsilon}(y))| \end{aligned}$$

$$:= I_1 + I_2 + I_3 \rightarrow 0.$$

Similar as above, by Hardy-Littlewood-Sobolev inequality and (2.11), $I_1, I_3 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Obviously,

$$|x - y| \geq \frac{\beta}{\varepsilon} \text{ if } (x, y) \in \bigcup_{i=1}^k B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{\beta}{\varepsilon}\right) \times \left(\mathbb{R}^N \setminus \bigcup_{i=1}^k B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right)\right).$$

Then

$$|I_2| \leq C(N, \alpha, \beta) \varepsilon^{N-\alpha} \left(\int_{\mathbb{R}^N} c |u_\varepsilon|^2 + \delta |u_\varepsilon|^{(N+\alpha)/(N-2)} \right)^2.$$

Noting that $\alpha \in ((N-4)_+, N)$, $(N+\alpha)/(N-2) \in (2, 2^*)$. Then we get $I_2 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Similarly, $\int_{\mathbb{R}^N} [I_\alpha * (F(u_{2,\varepsilon}))] F(u_{1,\varepsilon}) \rightarrow 0$ as $\varepsilon \rightarrow 0$. So we get (2.12).

By (2.12),

$$\begin{aligned} \Gamma_\varepsilon(u_\varepsilon) &= \Gamma_\varepsilon(u_{1,\varepsilon}) + \Gamma_\varepsilon(u_{2,\varepsilon}) + o_\varepsilon(1) \\ &\quad + \sum_{i=1}^k \int_{B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \setminus B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{\beta}{\varepsilon}\right)} \nabla \left(\varphi_\varepsilon \left(y - \frac{x_\varepsilon^i}{\varepsilon} \right) u_\varepsilon \right) \nabla \left(\left(1 - \varphi_\varepsilon \left(y - \frac{x_\varepsilon^i}{\varepsilon} \right) \right) u_\varepsilon \right) \\ &\quad + \sum_{i=1}^k \int_{B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \setminus B\left(\frac{x_\varepsilon^i}{\varepsilon}, \frac{\beta}{\varepsilon}\right)} V_\varepsilon(x) \varphi_\varepsilon \left(y - \frac{x_\varepsilon^i}{\varepsilon} \right) \left(1 - \varphi_\varepsilon \left(y - \frac{x_\varepsilon^i}{\varepsilon} \right) \right) |u_\varepsilon|^2 \\ &\geq \Gamma_\varepsilon(u_{1,\varepsilon}) + \Gamma_\varepsilon(u_{2,\varepsilon}) + o_\varepsilon(1). \end{aligned}$$

Therefore, we get (2.6).

Step 2. We claim that for $d, \varepsilon > 0$ small enough,

$$\Gamma_\varepsilon(u_{2,\varepsilon}) \geq \frac{1}{4} \|u_{2,\varepsilon}\|_\varepsilon^2. \quad (2.14)$$

Indeed,

$$\Gamma_\varepsilon(u_{2,\varepsilon}) \geq P_\varepsilon(u_{2,\varepsilon}) = \frac{1}{2} \|u_{2,\varepsilon}\|_\varepsilon^2 - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_{2,\varepsilon})) F(u_{2,\varepsilon}). \quad (2.15)$$

By (F1)-(F2), for any $\rho > 0$ there exists $c > 0$ (depending on ρ) such that $|F(t)| \leq \rho t^2 + c|t|^{(N+\alpha)/(N-2)}$ for $t \in \mathbb{R}$. Then by Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (I_\alpha * F(u_{2,\varepsilon})) F(u_{2,\varepsilon}) \right| &\leq C(N, \alpha) \left(\int_{\mathbb{R}^N} |F(u_{2,\varepsilon})|^{2N/(N+\alpha)} \right)^{(N+\alpha)/N} \\ &\leq C'(N, \alpha) \left[\int_{\mathbb{R}^N} (\rho^{2N/(N+\alpha)} |u_\varepsilon|^{4N/(N+\alpha)} + c |u_\varepsilon|^{2^*}) \right]^{(N+\alpha)/N} \\ &\leq C''(N, \alpha) \left(\rho^2 \|u_{2,\varepsilon}\|_{4N/(N+\alpha)}^2 + \|u_{2,\varepsilon}\|_{2^*}^{2(N+\alpha)/(N-2)} \right). \end{aligned}$$

Notice that $4N/(N+\alpha) \in (2, 2^*)$ and $2(N+\alpha)/(N-2) > 2$. By Sobolev's inequality,

$$\left| \int_{\mathbb{R}^N} (I_\alpha * F(u_{2,\varepsilon})) F(u_{2,\varepsilon}) \right| \leq C'''(N, \alpha) \left(\rho^2 \|u_{2,\varepsilon}\|_\varepsilon^2 + \|u_{2,\varepsilon}\|_\varepsilon^{2(N+\alpha)/(N-2)} \right)$$

Since $\{u_\varepsilon\}$ is bounded, we deduce from (2.5) that $\|u_{2,\varepsilon}\|_\varepsilon \leq 4d$ for sufficiently small $\varepsilon > 0$. Thus, taking d and ρ small enough, we have

$$\Gamma_\varepsilon(u_{2,\varepsilon}) \geq \frac{1}{4}\|u_{2,\varepsilon}\|_\varepsilon^2.$$

Step 3. For each $i = 1, 2, \dots, k$, we define

$$u_{1,\varepsilon}^i(x) = \begin{cases} u_{1,\varepsilon}(x), & x \in O_\varepsilon^i, \\ 0, & x \notin O_\varepsilon^i, \end{cases} \quad (2.16)$$

and set $W_\varepsilon^i(x) = u_{1,\varepsilon}^i\left(x + \frac{x_\varepsilon^i}{\varepsilon}\right)$. Then for fixed $i \in \{1, 2, \dots, k\}$, we can assume, up to a subsequence that as $\varepsilon \rightarrow 0$,

$$W_\varepsilon^i \rightharpoonup W^i \text{ weakly in } H^1(\mathbb{R}^N),$$

and W^i is a solution of

$$-\Delta W^i + V(x^i)W^i = (I_\alpha * F(W^i))f(W^i), \quad x \in \mathbb{R}^N.$$

In the following, we prove that $W_\varepsilon^i \rightarrow W^i$ strongly in H_ε . First, we prove that $W_\varepsilon^i \rightarrow W^i$ strongly in $L^p(\mathbb{R}^N)$ for any $p \in (2, 2^*)$. Otherwise, there exist $x_\varepsilon \in \mathbb{R}^N$ and $R > 0$ such that

$$\liminf_{\varepsilon \rightarrow 0} \int_{B(x_\varepsilon, R)} |W_\varepsilon^i - W^i|^2 > 0.$$

Obviously, $|x_\varepsilon| \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Let $z_\varepsilon = x_\varepsilon + x_\varepsilon^i/\varepsilon$, then

$$\liminf_{\varepsilon \rightarrow 0} \int_{B(z_\varepsilon, R)} |u_{1,\varepsilon}^i|^2 > 0. \quad (2.17)$$

Since $\varphi(x) = 0$ for $|x| \geq 2\beta$, $|x_\varepsilon| \leq 3\beta/\varepsilon$ (In fact $|x_\varepsilon| \leq 2\beta/\varepsilon$). Since $\varepsilon z_\varepsilon \in B(x_\varepsilon^i, 3\beta)$, we can assume $\varepsilon z_\varepsilon \rightarrow z^i \in O^i$ as $\varepsilon \rightarrow 0$.

Define $\widetilde{W}_\varepsilon^i(x) = u_{1,\varepsilon}^i(x + z_\varepsilon)$, then up to a subsequence that as $\varepsilon \rightarrow 0$,

$$\widetilde{W}_\varepsilon^i \rightharpoonup \widetilde{W}^i \not\equiv 0 \text{ weakly in } H^1(\mathbb{R}^N)$$

and \widetilde{W}^i satisfies

$$-\Delta \widetilde{W}^i + V(z^i)\widetilde{W}^i = (I_\alpha * F(\widetilde{W}^i))f(\widetilde{W}^i), \quad x \in \mathbb{R}^N.$$

Similar as in Step 1, we can get a contradiction. So $W_\varepsilon^i \rightarrow W^i$ strongly in $L^p(\mathbb{R}^N)$ for any $p \in (2, 2^*)$, which implies

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} (I_\alpha * F(W_\varepsilon^i))F(W_\varepsilon^i) = \int_{\mathbb{R}^N} (I_\alpha * F(W^i))F(W^i). \quad (2.18)$$

Then given any $i = 1, 2, \dots, k$, we deduce that

$$\begin{aligned}
\overline{\lim}_{\varepsilon \rightarrow 0} \Gamma_\varepsilon(u_{1,\varepsilon}^i) &\geq \overline{\lim}_{\varepsilon \rightarrow 0} P_\varepsilon(u_{1,\varepsilon}^i) \\
&= \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{\mathbb{R}^N} |\nabla W_\varepsilon^i|^2 + V_\varepsilon(x + x_\varepsilon^i/\varepsilon) |W_\varepsilon^i|^2 - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(W^i)) F(W^i) \\
&\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla W^i|^2 + V(x^i) |W^i|^2 - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(W^i)) F(W^i) \\
&= L_{V(x^i)}(W^i) \geq E_{m_i}.
\end{aligned} \tag{2.19}$$

Now, by the estimate (2.6), we get

$$\overline{\lim}_{\varepsilon \rightarrow 0} \left(\Gamma_\varepsilon(u_{2,\varepsilon}) + \sum_{i=1}^k \Gamma_\varepsilon(u_{1,\varepsilon}^i) \right) \leq \overline{\lim}_{\varepsilon \rightarrow 0} \Gamma_\varepsilon(u_\varepsilon) \leq E = \sum_{i=1}^k E_{m_i}. \tag{2.20}$$

On the other hand, by (2.14) and (2.19), by choosing $d > 0$ small enough,

$$\overline{\lim}_{\varepsilon \rightarrow 0} \left(\Gamma_\varepsilon(u_{2,\varepsilon}) + \sum_{i=1}^k \Gamma_\varepsilon(u_{1,\varepsilon}^i) \right) \geq \sum_{i=1}^k E_{m_i}. \tag{2.21}$$

Therefore, (2.20) and (2.21) imply that by choosing $d > 0$ small enough, for any $i = 1, 2, \dots, k$

$$\lim_{\varepsilon \rightarrow 0} \Gamma_\varepsilon(u_{1,\varepsilon}^i) = E_{m_i}. \tag{2.22}$$

By (2.14), $\|u_{2,\varepsilon}\|_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. By (2.19), we have $L_{V(x^i)}(W^i) = E_{m_i}$. Recalling that E_a is strictly increasing for $a > 0$, we obtain $x^i \in \mathcal{M}^i$ and $W^i(\cdot) = U_i(\cdot - z_i)$ for some $U_i \in S_{m_i}$ and $z_i \in \mathbb{R}^N$. Moreover, by (2.19) and (2.22), we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} |\nabla W_\varepsilon^i|^2 + V_\varepsilon(x + x_\varepsilon^i/\varepsilon) |W_\varepsilon^i|^2 = \int_{\mathbb{R}^N} |\nabla W^i|^2 + V(x^i) |W^i|^2.$$

Then $W_\varepsilon^i \rightarrow W^i$ strongly in $H^1(\mathbb{R}^N)$. Let $y_\varepsilon^i = z_i + x_\varepsilon^i/\varepsilon$, then $\varepsilon y_\varepsilon^i \rightarrow x^i \in \mathcal{M}^i$ and $u_{1,\varepsilon}^i \rightarrow \varphi_\varepsilon(\cdot - y_\varepsilon^i) U_i(\cdot - y_\varepsilon^i)$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \rightarrow 0$. Noting that $\text{supp}(u_{1,\varepsilon}^i) \subset O_\varepsilon^i$, $u_{1,\varepsilon}^i \rightarrow \varphi_\varepsilon(\cdot - y_\varepsilon^i) U_i(\cdot - y_\varepsilon^i)$ in H_ε as $\varepsilon \rightarrow 0$, which implies that

$$u_{1,\varepsilon} = \sum_{i=1}^k u_{1,\varepsilon}^i = \sum_{i=1}^k \varphi_\varepsilon(x - y_\varepsilon^i) U_i(x - y_\varepsilon^i) + o_\varepsilon(1) \text{ in } H_\varepsilon.$$

Since $\lim_{\varepsilon \rightarrow 0} \Gamma_\varepsilon(u_{2,\varepsilon}) = 0$, from (2.14), we know $u_{2,\varepsilon} \rightarrow 0$ in H_ε , then the proof is completed. \square

Immediately, as a consequence of Proposition 2.5, we have

Proposition 2.6. *For sufficiently small $d > 0$, there exist constants $\eta > 0$ and $\varepsilon_0 > 0$, such that $|\Gamma'_\varepsilon(u)| \geq \eta$ for $u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^d \setminus X_\varepsilon^{\frac{d}{2}})$ and $\varepsilon \in (0, \varepsilon_0)$.*

Now, we fix $d > 0$ such that Proposition 2.6 holds. Choose $R_0 > 0$ large enough such that $O \subset B(0, R_0)$ and $\gamma_\varepsilon(s) \in H_0^1(B(0, \frac{R}{\varepsilon}))$ for any $s \in T$, $R > R_0$.

Proposition 2.7. *Given $\varepsilon > 0$ sufficiently small, then there exists a sequence $\{u_n^R\} \subset X_\varepsilon^{\frac{d}{2}} \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, \frac{R}{\varepsilon}))$, such that $\lim_{n \rightarrow \infty} \|\Gamma'(u_n^R)\| = 0$ in $H_0^1(B(0, \frac{R}{\varepsilon}))$.*

Proof. The proof is similar to [6]. To the contrary, for $\varepsilon > 0$ small enough, there exists $a_R(\varepsilon) > 0$ such that $\|\Gamma'_\varepsilon(u)\| \geq a_R(\varepsilon)$ for any $u \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, \frac{R}{\varepsilon}))$. It follows from Proposition 2.4 that there exists $\alpha_0 \in (0, E - \tilde{E})$ such that if $\varepsilon > 0$ small enough and $\Gamma_\varepsilon(\gamma_\varepsilon(s)) \geq D_\varepsilon - \alpha_0$, then $\gamma_\varepsilon(s) \in X_\varepsilon^{\frac{d}{2}} \cap H_0^1(B(0, \frac{R}{\varepsilon}))$. Thus, by a deformation argument in $H_0^1(B(0, \frac{R}{\varepsilon}))$, there exist a $\mu_0 \in (0, \alpha_0)$ and a path $\gamma \in (C[0, T], H_\varepsilon)$ such that

$$\gamma(s) \begin{cases} = \gamma_\varepsilon(s) & \text{for } \gamma_\varepsilon(s) \in \Gamma_\varepsilon^{D_\varepsilon - \alpha_0} \\ \in X_\varepsilon^d & \text{for } \gamma_\varepsilon(s) \notin \Gamma_\varepsilon^{D_\varepsilon - \alpha_0}, \end{cases}$$

and

$$\Gamma_\varepsilon(\gamma(s)) < D_\varepsilon - \mu_0, s \in T. \quad (2.23)$$

Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be such that $\psi(x) = 1$ for $x \in O^\delta$, $\psi(x) = 0$ for $x \notin O^{2\delta}$, $\psi(x) \in [0, 1]$ and $|\nabla \psi| \leq \frac{2}{\delta}$. For $\gamma(s) \in X_\varepsilon^d$, we define $\gamma_1(s) = \psi_\varepsilon \gamma(s)$, $\gamma_2(s) = (1 - \psi_\varepsilon)\gamma(s)$, where $\psi_\varepsilon = \psi(\varepsilon x)$, then

$$\begin{aligned} \Gamma_\varepsilon(\gamma(s)) &= \Gamma_\varepsilon(\gamma_1(s)) + \Gamma_\varepsilon(\gamma_2(s)) + Q_\varepsilon(\gamma(s)) - Q_\varepsilon(\gamma_1(s)) - Q_\varepsilon(\gamma_2(s)) \\ &\quad + \int_{\mathbb{R}^N} (\psi_\varepsilon(1 - \psi_\varepsilon)|\nabla \gamma(s)|^2 + V_\varepsilon \psi_\varepsilon(1 - \psi_\varepsilon)|\gamma(s)|^2) + o_\varepsilon(1) \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(\gamma(s)))F(\gamma(s)) + \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(\gamma_1(s)))F(\gamma_1(s)) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(\gamma_2(s)))F(\gamma_2(s)). \end{aligned}$$

Notice that

$$\begin{aligned} Q_\varepsilon(\gamma(s)) &= \left(\int_{\mathbb{R}^N} \chi_\varepsilon |\gamma_1(s)|^2 + \int_{\mathbb{R}^N} \chi_\varepsilon |\gamma_2(s)|^2 - 1 \right)_+^2 \\ &\geq \left(\int_{\mathbb{R}^N} \chi_\varepsilon |\gamma_1(s)|^2 - 1 \right)_+^2 + \left(\int_{\mathbb{R}^N} \chi_\varepsilon |\gamma_2(s)|^2 - 1 \right)_+^2 \\ &= Q_\varepsilon(\gamma_1(s)) + Q_\varepsilon(\gamma_2(s)). \end{aligned}$$

Then we get

$$\begin{aligned} \Gamma_\varepsilon(\gamma(s)) &\geq \Gamma_\varepsilon(\gamma_1(s)) + \Gamma_\varepsilon(\gamma_2(s)) + \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(\gamma_2(s)))F(\gamma_2(s)) \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(\gamma(s)))F(\gamma(s)) + \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(\gamma_1(s)))F(\gamma_1(s)) + o_\varepsilon(1). \end{aligned} \quad (2.24)$$

Since $Q_\varepsilon(\gamma(s))$ is uniformly bounded with respect to ε , there exists $C > 0$ such that

$$\int_{\mathbb{R}^N \setminus O_\varepsilon} |\gamma(s)|^2 \leq C\varepsilon^\mu, s \in T. \quad (2.25)$$

Let $H(\gamma(s)) := F(\gamma(s)) - F(\gamma_1(s)) - F(\gamma_2(s))$, then

$$H(\gamma(s))(x) = 0, \text{ if } x \notin O_\varepsilon^{2\delta} \setminus O_\varepsilon^\delta.$$

Then similar to (2.13), for any $\delta_0 > 0$, by virtue of the Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} (I_\alpha * H(\gamma(s))) F(\gamma(s)) \right| \\ & \leq C'(N, \alpha) \left[\int_{O_\varepsilon^{2\delta} \setminus O_\varepsilon^\delta} (c|\gamma(s)|^{4N/(N+\alpha)} + \delta_0^{2N/(N+\alpha)} |\gamma(s)|^{2*}) \right]^{(N+\alpha)/(2N)}. \end{aligned}$$

Noting that $\gamma(s)$ is uniformly bounded in $H^1(\mathbb{R}^N)$ for ε and $s \in T$, thanks to the interpolation inequality and (2.25), we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} [I_\alpha * H(\gamma(s))] F(\gamma(s)) = 0.$$

Similarly, $i \neq j$ and $i, j = 1, 2$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} [I_\alpha * F(\gamma_i(s))] H(\gamma(s)) = 0, \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} [I_\alpha * F(\gamma_i(s))] F(\gamma_j(s)) = 0,$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} [I_\alpha * F(\gamma_2(s))] F(\gamma_2(s)) = 0.$$

Then

$$\int_{\mathbb{R}^N} [I_\alpha * F(\gamma(s))] F(\gamma(s)) = \int_{\mathbb{R}^N} [I_\alpha * F(\gamma_1(s))] F(\gamma_1(s)) + o_\varepsilon(1).$$

By (2.24),

$$\Gamma_\varepsilon(\gamma(s)) \geq \Gamma_\varepsilon(\gamma_1(s)) + o_\varepsilon(1). \quad (2.26)$$

For $i = 1, 2, \dots, k$, let

$$\gamma_1^i(s)(x) = \begin{cases} \gamma_1(s)(x), & \text{for } x \in (O_\varepsilon^i)^{2\delta}, \\ 0, & \text{for } x \notin (O_\varepsilon^i)^{2\delta}, \end{cases}$$

then

$$\Gamma_\varepsilon(\gamma_1(s)) \geq \sum_{i=1}^k \Gamma_\varepsilon(\gamma_1^i(s)) = \sum_{i=1}^k \Gamma_\varepsilon^i(\gamma_1^i(s)). \quad (2.27)$$

Since $0 < \alpha_0 < E - \tilde{E}$, by Proposition 2.4, for all $i \in \{1, 2, \dots, k\}$, $\gamma_1^i(s) \in \Phi_\varepsilon^i$. Thus, thanks to [43, Proposition 3.4] and (2.27), we deduce that

$$\max_{s \in T} \Gamma_\varepsilon(\gamma(s)) \geq E + o_\varepsilon(1).$$

Combining with (2.23), we get $E \leq D_\varepsilon - \mu_0$, which is a contradiction. \square

Proposition 2.8. *Given $\varepsilon, d > 0$ sufficiently small, Γ_ε has a nontrivial critical point $u \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$.*

Proof. Let $\{u_n^R\}$ be a Palais-Smale sequence of Γ_ε obtained above, then due to $u_n^R \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$, $\{u_n^R\}$ is uniformly bounded in $H_0^1(B(0, \frac{R}{\varepsilon}))$ for n . Up to a subsequence, as $n \rightarrow \infty$, $u_n^R \rightarrow u_\varepsilon^R$ strongly in $H_0^1(B(0, \frac{R}{\varepsilon}))$ and u_ε^R is a critical point of Γ_ε on $H_0^1(B(0, \frac{R}{\varepsilon}))$ and satisfies

$$-\Delta u_\varepsilon^R + \tilde{V}_\varepsilon u_\varepsilon^R = [I_\alpha * F(u_\varepsilon^R)] f(u_\varepsilon^R), \quad |x| \leq R/\varepsilon, \quad (2.28)$$

where

$$\tilde{V}_\varepsilon = V_\varepsilon + 4 \left(\int_{\mathbb{R}^N} \chi_\varepsilon |u_\varepsilon^R|^2 dx - 1 \right)_+ \chi_\varepsilon.$$

Since $f(t) = 0$ for $t < 0$, one knows $u_\varepsilon^R \geq 0$ in $B(0, R/\varepsilon)$. We extend $u_\varepsilon^R \in H_0^1(B(0, R/\varepsilon))$ to $u_\varepsilon^R \in H_\varepsilon$ by zero outside $B(0, R/\varepsilon)$. Noting that $u_\varepsilon^R \in X_\varepsilon^R$, $\{u_\varepsilon^R\}$ is uniformly bounded in H_ε for R, ε . By repeating the argument in [28, Proposition 3.1], for any $p \in [2, \frac{N}{\alpha} \frac{2N}{N-2})$, there exists C_p (independent of ε, R) such that $\|u_\varepsilon^R\|_p \leq C_p \|u_\varepsilon^R\|_2$. Then similar as in Proposition 2.2, we know $I_\alpha * F(u_\varepsilon^R)$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$ for ε, R . So there exists C (independent of ε, R) such that

$$-\Delta u_\varepsilon^R + u_\varepsilon^R \leq C f(u_\varepsilon^R), \quad |x| \leq R/\varepsilon.$$

Thanks to $(N+\alpha)/(N-2) < 2^*$, it follows from the standard Moser iteration [18] that $\{u_\varepsilon^R\}$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$ for ε, R . On the other hand, by $u_n^R \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$, there exists $C > 0$ (independent of ε, n, R) such that

$$\int_{\mathbb{R}^N} \chi_\varepsilon |u_n^R|^2 dx \leq C, \quad n \in \mathbb{N}.$$

By Fatou's Lemma, $\int_{\mathbb{R}^N \setminus O_\varepsilon} |u_\varepsilon^R|^2 dx \leq C \varepsilon^\mu$ for all ε, R . Then it follows from [18, Theorem 8.17] and the comparison principle that there exist $c, C > 0$ (independent of ε, R) such that

$$u_\varepsilon^R(x) \leq C \exp(-c|x|), \quad x \in \mathbb{R}^N, \quad (2.29)$$

which yields that, up to a subsequence, $u_\varepsilon^R \rightarrow u_\varepsilon$ strongly in $L^p(\mathbb{R}^N)$ as $R \rightarrow \infty$ for any $p \in [2, 2^*)$. Thus, $u_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$ is a nontrivial critical point of Γ_ε . Obviously, $0 \notin X_\varepsilon^d$ if $d > 0$ small enough. So $u_\varepsilon \not\equiv 0$ if $d > 0$ small. \square

2.3. Completion of the proof for Theorem 1.1.

Proof. By Proposition 2.8, there exist $d > 0$ and $\varepsilon_0 > 0$, such that Γ_ε has a nontrivial critical point $u_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$ for $\varepsilon \in (0, \varepsilon_0)$. Since $u_\varepsilon^R \rightarrow u_\varepsilon$ strongly in $L^2(\mathbb{R}^N)$ as $R \rightarrow \infty$ and (2.29), there exists $C > 0$ such that $\sup_{\varepsilon \in (0, \varepsilon_0)} \|u_\varepsilon\|_\infty \leq C$. By (F1) and $u_\varepsilon \not\equiv 0$,

$$\inf_{\varepsilon \in (0, \varepsilon_0)} \|u_\varepsilon\|_\infty \geq \rho. \quad (2.30)$$

Since $f(t) = 0$ for $t \leq 0$, we see that $u_\varepsilon \geq 0$. By (2.30) and the weak Harnack inequality (see [18]), $u_\varepsilon > 0$ in \mathbb{R}^N . By Proposition 2.5, there exist $\{y_\varepsilon^i\}_{i=1}^k \subset \mathbb{R}^N, x^i \in \mathcal{M}^i, U_i \in S_{m_i}$ such that for any $1 \leq i \leq k$,

$$\lim_{\varepsilon \rightarrow 0} |\varepsilon y_\varepsilon^i - x^i| = 0 \text{ and } \lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - \sum_{i=1}^k U_i(\cdot - y_\varepsilon^i)\|_\varepsilon = 0.$$

Let $w_\varepsilon^i(y) = u_\varepsilon(y + y_\varepsilon^i)$, then $\lim_{\varepsilon \rightarrow 0} \|w_\varepsilon^i - U_i\|_2 = 0$, which implies that for any $\sigma > 0$, there exists $R > 0$ (independent of ε, i) such that

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \int_{\mathbb{R}^N \setminus B(0, R)} (w_\varepsilon^i)^2 \leq \sigma.$$

Similar as above, $I_\alpha * F(u_\varepsilon)$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$ for ε . Then it follows from [18, Theorem 8.17] and the comparison principle that for each $1 \leq i \leq k$, there exist $M > 0$ (independent of ε, i) and $y_\varepsilon^i \in \mathbb{R}^N$, such that

$$0 < w_\varepsilon^i(y) \leq M \exp\left(-\frac{|y|}{2}\right) \text{ for } y \in \mathbb{R}^N, \varepsilon \in (0, \varepsilon_0).$$

Then

$$0 < u_\varepsilon(y) \leq M \exp\left(-\frac{1}{2} \min_{1 \leq i \leq k} |y - y_\varepsilon^i|\right) \text{ for } y \in \mathbb{R}^N, \varepsilon \in (0, \varepsilon_0), \quad (2.31)$$

which yields that $Q_\varepsilon(u_\varepsilon) = 0$ for small $\varepsilon > 0$. Therefore, u_ε is a critical point of P_ε . This completes the proof. \square

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(M. B. Yang)
DEPARTMENT OF MATHEMATICS
ZHEJIANG NORMAL UNIVERSITY
JINHUA 321004, PR CHINA
E-mail address: mbyang@zjnu.edu.cn

(J. J. Zhang)
COLLEGE OF MATHEMATICS AND STATISTICS
CHONGQING JIAOTONG UNIVERSITY
CHONGQING 400074, PR CHINA
E-mail address: zhangjianjun09@tsinghua.org.cn

(Y. M. Zhang)
WUHAN INSTITUTE OF PHYSICS AND MATHEMATICS
CHINESE ACADEMY OF SCIENCES
WUHAN 430071, CHINA
E-mail address: zhangym802@126.com